Journal of Applied Mechanics and Technical Physics, Vol. 39, No. 1, 1998

SUBCRITICAL FLOWS FROM BENEATH A SHIELD

V. I. Nalimov

UDC 532.59

The problem of a two-dimensional stationary flow of an ideal incompressible heavy vortex-free liquid that flows from beneath a shield is studied. The bottom is considered smooth and horizontal, and the flow is assumed to be subcritical. The existence of a solution that is different from a uniform flow is proved in an exact formulation. It is shown that the solution behaves like a Nekrasov wave at infinity.

We analyze the problem of a two-dimensional steady-state flow of an ideal incompressible vortex-free liquid that flows from beneath a plane horizontal shield. The bottom is assumed to be smooth and horizontal. The flow is assumed to be subcritical, i.e., the characteristic velocity of the liquid flow is lower than the propagation velocity of long waves of infinitely small amplitude.

The existence of traveling waves for an infinitely deep liquid was proved independently by A. I. Nekrasov in 1921 and Levi-Civita in 1925. A little later, J. Struik, followed by Nekrasov, formulated similar theorems for a liquid of finite depth. The relevant references were given by J. Stoker in [1]. The author proved the existence of subcritical flows over an uneven bottom in [2]. Maklakov [3] found the problem of subscritical flow about a vortex to be solvable.

If we suppose that a free boundary and a solid wall (a shield) are in contact, the nonlinear boundary conditions are greatly complicated. The branching problem that arises in solving the problem of subcritical flow from beneath a shield has a troublesome unpleasant feature, namely, the number of solvability conditions is larger than the number of free parameters (the effect of infinity). Nevertheless, one manages to prove the solvability of this wave problem if the symmetry of the problem is used and a new parameter — the wave phase — is introduced.

1. Formulation of the Problem. To describe liquid flows, the dimensionless velocity potential φ and stream function ψ are chosen as independent variables [1]. This choice of independent variables allows one to work in a fixed domain between the two streamlines $\psi = 0$ and $\psi = 1$, rather than in a partially unknown flow region.

As is known, the complex velocity potential $\chi = \varphi + i\psi$ is an analytic function of the independent complex variable $z = x + iy$. The conjugate complex velocity $\bar{w} = d\chi/dz$ is an analytic function of z as well. With $w = \exp\{-i(\theta + i\tau)\}$, the flow problem for a liquid is reduced [1] to searching for the analytic function $\theta + i\tau$ of χ with the boundary condition $\theta_\psi - \lambda \exp\{-3\tau\} \sin\theta = 0$ on the "free surface" $\psi = 1$, $\varphi > 0$ with constant $\lambda = gh_0U^{-2}$ in a unit horizontal band, where g is the acceleration of gravity, h_0 is the characteristic depth of the flow, and U is the characteristic flow velocity. It is assumed without loss of generality that the point of contact of the free surface and the lid transforms into the point $\varphi = 0$, $\psi = 1$. The requirement of flow subcriticality means that $\lambda > 1$. Since the angle of inclination of the flow velocity at the bottom and at the lid should coincide with that of the tangent, then $\theta = 0$ for $\psi = 0$, $\psi = 1$ ($\varphi \leqslant 0$).

It is assumed that the flow to be found differs little from a uniform flow: $\theta = \varepsilon^2 U$, $\tau = \varepsilon^2 V$, $\lambda = \lambda_0 + \varepsilon^2\lambda_1$, where $\lambda_0 > 1$, ε is a small parameter, and λ_1 is a sought parameter. We use x and y as the

Lavrent'ev Institute of Hydrodynamics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 39, No. 1, pp. 54–60, January–February, 1998. Original article submitted July 2, 1996.

independent variables instead of φ and ψ. In addition, we assume that

$$f_0(U, V, \lambda_1, \varepsilon) = \lambda_1 U - 3\lambda_0 UV + \varepsilon^2 f_1(U, V, \lambda_1, \varepsilon); \tag{1.1}$$

$$f_1(U, V, \lambda_1, \varepsilon) = \varepsilon^{-6}(\lambda_0 + \varepsilon\lambda_1)[\exp(-3\varepsilon^2 U) - 1 + 3\varepsilon^2 V]\sin(\varepsilon^2 U)$$
$$+\varepsilon^{-6}(\lambda_0 + \varepsilon\lambda_1)[\sin(\varepsilon^2 U) - \varepsilon^2 U](1 - 3\varepsilon^2 V) - 3\lambda_1 UV. \tag{1.2}$$

In the new notation, the initial problem is formulated as a problem of searching in the domain $-\infty < x < \infty$, $0 < y < 1$ for a pair of functions (U, V) from the Cauchy–Riemann system

$$U_y + V_x = 0, \quad U_x - V_y = 0 \tag{1.3}$$

with the boundary conditions at the "solid walls"

$$U = 0 \quad (y = 0); \qquad U = 0 \quad (y = 1,\ x \leqslant 0) \tag{1.4}$$

and the condition at the "free boundary"

$$U_y - \lambda_0 U = \varepsilon^2 f_0(U, V, \lambda_1, \varepsilon) \qquad (y = 1,\ x > 0). \tag{1.5}$$

Since the function V, which is conjugate to U, is found with accuracy up to an arbitrary constant, an additional condition is required to define this function uniquely. It is assumed that

$$U(x, y) \to 0 \qquad (x \to -\infty). \tag{1.6}$$

With the use of the prescribed function $u(x) = U(x, 1)$ the functions U and V are recovered uniquely from the Cauchy–Riemann system (1.3) with the boundary condition $U(x, 0) = 0$ and the condition at infinity (1.6). In view of this, in what follows, the function $u(x)$ [or the pair $u(x)$ and $v(x) = V(x, 1)$] will be called a solution of problem (1.3)–(1.6).

2. Functional Spaces. The main obstacle in the investigation of problem (1.3)–(1.6) is the train of nonlinear waves formed for subcritical liquid flows behind the shield. Therefore, it is natural to study this problem in functional spaces related to it. Let ω be a positive root of the equation $\xi\coth\xi - \lambda_0 = 0$ and ρ_0 be the smallest positive root of the equation $\xi\cot\xi - \lambda_0 = 0$; the number ρ is chosen such that $0 < \rho < \rho_0$.

The set of functions that exponentially decrease with ρ, are defined on the entire R axis, and have the finite norm

$$\|u\|_{E^s} = \max_{\tau=\pm\rho} \|u(x)\exp(\tau x)\|_{H^s(R)}$$

is denoted by E^s, and $\overset{\circ}{E}{}^s$ is the subspace of functions from E^s that are equal to zero for $x < 0$. Like the properties of the spaces $H^s(0, L)$, the basic properties of the Sobolev–Shvarts spaces $H^s(R)$, which are periodic along the axis of functions, can be found in [4, 5].

Let the function $\theta(x)$ possess the following properties: $\theta' \in \overset{\circ}{C}{}^\infty(R)$, $\theta = 0$ for $x \leqslant 1$, and $\theta(x) = 1$ for $x \geqslant 2$. The functional spaces V^s, in which problem (1.3)–(1.6) is studied, consist of functions of the form $u = u^0 + \theta u^+$, $u^0 \in H^s(R)$, and $u^+ \in H^s(0, L)$, the latter being a periodic function with period $L = 2\pi/\omega$. The norm in the space V^s is defined by the equality $\|u\|_{V^s} = \|u^0\|_{E^s} + \|u^+\|_{H^s(0,L)}$. Clearly, V^s is a complete linear normalized space.

For $s > 1/2$, the space V^s is a Banach algebra [6]: $\|uv\|_{V^s} \leqslant C_0\|u\|_{V^s}\|v\|_{V^s}$. This makes it possible to estimate the norms of composite functions in this space. If the series $\sum t^\alpha F_\alpha = F(t)$ $(t = t_1, \dots, t_m)$ converges in a neighborhood of the coordinate origin, for a composite function at $s > 1/2$ we have

$$\|F(\mathrm{v})\|_{V^s} \leqslant \tilde{F}(\|\mathrm{v}\|_{V^s}), \quad \|\mathrm{v}\|_{V^s} = \sum_{k=1}^{m} \|v_k\|_{V^s}, \tag{2.1}$$

where

$$\tilde{F}(X) = \sum_{k=1}^{\infty} \frac{1}{k!}\, C_0^k \max_{|\alpha|=k} (\alpha!|F_\alpha|)X^k.$$

For convergence of the series for the majorants, it is naturally assumed that the norm of the vector function v is small. Because the majorant of a derivative does not exceed the derivative of the majorant, at $s > 1/2$ the formula

$$F(v) - F(w) = \sum_{j=1}^{m}(v_j - w_j)\int_0^1 F_{t_j}(\xi v + (1-\xi)w)\,d\xi \tag{2.2}$$

leads to the estimate

$$\|F(v) - F(w)\|_{V^s} \leqslant \|(v-w)\|_{V^s}\int_0^1 \tilde{F}'(\xi\|v\|_{V^s} + (1-\xi)\|w\|_{V^s})\,d\xi. \tag{2.3}$$

Let the projection operators Π_+ and Π_0 act according to the rules $\Pi_+ u = u^+$ and $\Pi_0 u = u^0$. For the composite function

$$\Pi_+ F(v) = F(v^+), \quad \Pi_0 F(v) = F(v) - F(\theta v^+) + (1-\theta)F(\theta v^+) + \theta(F(\theta v^+) - F(v^+))$$

and for $s > 1/2$, the estimate

$$\|\Pi_0 F(v)\|_{E^s} \leqslant C\tilde{F}'(C\|v\|_{V^s}) \tag{2.4}$$

holds. Here and below, various insignificant constants are denoted by the same symbol C.

The representation $F(v) = \Pi_0 F(v) + \theta F(v^+)$ is checked in a very simple way. The function $\theta(1-\theta) \in \overset{\circ}{C}{}^\infty(R)$. Since the product of a function from E^s and a function from V^s at $s > 1/2$ belongs to the space E^s [6] with the corresponding estimate, the proposition (2.4) follows from formulas (2.2) and (2.3).

For functions specified on the real axis, the shear operator T_ν is defined by the equality $T_\nu u(x) = u(x-\nu)$. It is clear that $T_\nu F(\mathrm{v}) = F(T_\nu \mathrm{v})$ for any composite function and

$$\|T_\nu u\|_{H^s(0,L)} = \|u\|_{H^s(0,L)}, \quad \|\partial T_\nu u/\partial\nu\|_{H^s(0,L)} \leqslant \omega\|u\|_{H^{s+1}(0,L)} \tag{2.5}$$

for any periodic function.

3. Linear Problem. The author [6] studied the linear problem (1.3)–(1.6) in the spaces V^s, with a given function $f(x)$ in the right-hand side of the boundary condition (1.5). The linear homogeneous problem (1.3)–(1.5) has a nontrivial solution (Φ, Ψ). In accordance with the agreement in Sec. 1, the pair φ and ψ [$\varphi = \Phi(x,1)$ and $\psi = \Psi(x,1)$] is called an eigenfunction of problem (1.3)–(1.5). In [6], the representation $\varphi = \varphi^0 + \theta\sin\omega(x - x_0)$ was established and it was shown that $\varphi^0 \in E^s$ for any $s < 1$. The function ψ, which is conjugate to φ and which meets the requirement (1.6), belongs to the space V^s as well [6], and its periodic component is calculated by the formula

$$\psi^+ = \coth\omega\cos\omega(x - x_0) - \int_0^\infty \varphi^0(x)\,dx - \frac{1}{\omega}\int_0^\infty \theta'(x)\cos\omega(x-x_0)\,dx.$$

If the three orthogonality conditions

$$\int_0^L f^+(x)\,dx = 0; \tag{3.1}$$

$$\int_0^L f^+(x)\cos\omega x\,dx = 0, \quad \int_0^L f^+(x)\sin\omega x\,dx = 0 \tag{3.2}$$

are satisfied, a solution of the linear problem (1.3)–(1.6) exists. The integral representation of this solution contains the operators K_1 and K_2 with the symbols

$$\hat{K}_1(\xi) = \sqrt{\pi - i\xi}/Y^+(\xi), \quad \hat{K}_2(\xi) = Y^-(\xi)\sqrt{\pi + i\xi}/(\xi^2 - \omega^2).$$

The caret operator sets the Fourier transform $\hat{u}(\xi)$ in correspondence with the generalized function $u(x)$, and the functions $Y^{\pm}(\xi)$ possess the following properties for all real ξ:

$$C^{-1} \leqslant |Y^{\pm}(\xi)| \leqslant C_1, \quad Y^{\pm}(\xi) = \overline{Y^{\pm}(\xi)}, \quad \hat{K}_1(\xi)\hat{K}_2(\xi) = \hat{m}(\xi), \qquad \hat{m}(\xi) = (\xi\coth\xi - \lambda_0)^{-1},$$

where the overscore refers to the operation of complex conjugation. In what follows, the convolution operator that corresponds to $\hat{m}(\xi)$ is denoted by M.

The solution of the linear problem (1.3)–(1.5) admits the representation $u = Kf + C\varphi$ and $v = K_0 u + C_1$ with arbitrary constants C and C_1. Here and below, the operator K_0 is a convolution operator with the symbol $\hat{K}_0(\xi) = i\coth\xi$, and $K = K_2 H K_1$. Here $H(x)$ is the Heaviside function: $H(x) = 0$ for $x < 0$ and $H(x) = 1$ for $x \geqslant 1$.

The operator K_0 is defined on functions from V^s that are subject to the orthogonality condition (3.1). If $s < 0$, this condition is understood in the sense of generalized functions. For such functions, we have

$$\Pi_+ K_0 f = K_0 f^+ - a(f), \qquad a(f) = \hat{f}(0); \tag{3.3}$$

$$\|K_0 f^+\|_{H^s(0,L)} + \|\Pi_0 K_0 f\|_{E^s} \leqslant C\|f\|_{V^s}. \tag{3.4}$$

The real linear functional $a(f)$ is continuous on functions subject to the orthogonality condition (3.1): for any r, we have

$$|a(f)| \leqslant C\|f\|_{V^r}. \tag{3.5}$$

For $0 < s < 1$, the operator K acts from the subspace of functions from V^s that are subject to the orthogonality condition (3.2) to the space V^{s+1}. The equality $\Pi_+ K f = M f^+ - \mathrm{Im}\{b(f)\exp(i\omega x)\}$ holds. The complex linear functional $b(f) = (1/2\omega)Y^-(\omega)\sqrt{\pi + i\omega}\widehat{HK}_1(\omega)$ is defined on functions $f \in V^r$, $r > -1/2$, that are subject to the orthogonality condition (3.1):

$$|b(f)| \leqslant C\|f\|_{V^r}. \tag{3.6}$$

In addition, the estimate

$$\|Mf^+\|_{H^{s+1}(0,L)} + \|\Pi_0 K f\|_{E^{s+1}} \leqslant C\|f\|_{V^s} \tag{3.7}$$

holds for the indicated s.

4. Nonlinear Problem of Flow from beneath a Shield. According to the aforesaid, the function that is conjugate to u and is subject to the orthogonality condition (3.1) is of the form $v = K_0 u$. Therefore, problem (1.3)–(1.6) has a nontrivial solution if this solution exists for the equation

$$u = \varepsilon^2 K\{\lambda_1 u + f_0(u, K_0 u, \lambda_1, \varepsilon)\} + \varphi. \tag{4.1}$$

Let $\nu = x_0 + \varepsilon\lambda_3$ and $w(x) = w^0(x) + \theta(x) w^+(x - \nu)$, where

$$w^+(x) = \sum_{n=2}^{\infty} a_n \sin\omega x. \tag{4.2}$$

The solution of Eq. (1.4) is sought for in the form

$$u(x) = \varphi^0(x) + (1 + \varepsilon\lambda_2)\theta(x)\sin\omega(x - \nu) + \varepsilon w(x). \tag{4.3}$$

The sought quantities will be the vector parameter $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ and the vector function $\mathrm{w}(x) = (w^+(x), w^0(x))$. For a function of the form (4.3), we have

$$a(u) = -a_0 - \varepsilon\Lambda_0(\mathrm{w}, \boldsymbol{\lambda}, \varepsilon), \tag{4.4}$$

where $a_0 = -\hat{\varphi}(0) = C_2\sqrt{\pi}\omega^{-2}Y^-(0)$ and

$$\Lambda_0 = -a(w) + \mathrm{Re}\{\hat{\theta}'(\omega)e^{i\omega\nu}[\lambda_2 + \varepsilon^{-1}(-e^{-i\omega\lambda_2} + 1)]\}. \tag{4.5}$$

In [6], the constant C_2 was found from the normalization condition $\varphi^+ = \sin\omega(x - x_0)$. According to the representation (4.3) of the solution and formula (3.3), we have

$$v = v^0 + \theta(x)[a_0 + \varepsilon\Lambda_0 + \coth\omega(1 + \varepsilon\lambda_2)\cos\,\omega(x - \nu) + \varepsilon K_0 w^+(x - \nu)], \tag{4.6}$$

where $v^0 = \Pi_0 K_0 u$. By the definition of the shear operator, we have

$$T_{-\nu} u^+ = (1 + \varepsilon\lambda_2) \sin \omega x + \varepsilon w^+(x).$$

The shear operator commutes with any convolution operator. Therefore, $T_{-\nu} v^+ = a + \varepsilon\Lambda_0 + (1 + \varepsilon\lambda_2) \coth \omega \cos \omega x + \varepsilon K_0 w^+(x)$.

According to the definition of the operator K_0, we have

$$K_0 w^+(x) = \sum_{n=2}^{\infty} a_n \coth n\omega \cos n\omega x.$$

Therefore, the functions $f_k^+(\mathbf{w}, \boldsymbol{\lambda}, \varepsilon) = T_{-\nu}\Pi_+ f_k(u, v, \varepsilon) = f_k(T_{-\nu}u^+, T_{-\nu}v^+, \varepsilon)$ $(k = 0, 1)$ are odd in x. The function $f_0(u, v, \lambda_1, \varepsilon)$ will be subject to the orthogonality conditions (3.1) and (3.2) if

$$\int_0^L f_0^+(\mathbf{w}, \boldsymbol{\lambda}, \varepsilon) \sin \omega x \, dx = 0.$$

This equality determines the parameter

$$\lambda_1 = 3a_0 + \varepsilon\Lambda_1(\mathbf{w}, \boldsymbol{\lambda}, \varepsilon) \tag{4.7}$$

with the functional

$$\Lambda_1 = 3\Lambda_0 + \frac{3}{2}\lambda_0(\coth\omega - \coth 2\omega)a_2 - \frac{\varepsilon}{\pi(1+\varepsilon\lambda_2)} \int_0^L [3\lambda_0 w^+ K_0 w^+ - f_1^+(\mathbf{w}, \boldsymbol{\lambda}, \varepsilon)] \sin\omega x \, dx. \tag{4.8}$$

Let u and v be specified by equalities (4.3) and (4.6), respectively, and $f(\mathbf{w}, \boldsymbol{\lambda}, \varepsilon) = f_0(u, v, \lambda_1, \varepsilon)$. By definition, we have $T_{-\nu} f^+ = f_0^+$. In addition, let $\Lambda_4(\mathbf{w}, \boldsymbol{\lambda}, \varepsilon) = b(f(\mathbf{w}, \boldsymbol{\lambda}, \varepsilon))$. By virtue of the choice of the parameter λ_1 of (4.7) and the operator K of (3.3), Eq. (4.1) leads to the equality for lower harmonics

$$(1 + \varepsilon\lambda_2)\sin\omega x = \sin\omega(x - x_0 + \nu) - \varepsilon^2 \mathrm{Im}\,\{e^{i\omega(x+\nu)}\Lambda_4\}.$$

This equality determines the parameters λ_2 and λ_3:

$$\lambda_k = \varepsilon\Lambda_k(\mathbf{w}, \boldsymbol{\lambda}, \varepsilon) \qquad (k = 2, 3).$$

Here $\Lambda_2 = \sin\omega\nu \mathrm{Im}\,\lambda_4 - \cos\omega\nu \mathrm{Re}\,\lambda_4 - \varepsilon^{-2}[1 - \cos(\varepsilon\omega\lambda_3)]$, $\Lambda_3 = \varepsilon^{-1}\omega^{-1}\arcsin\{\varepsilon^2[\sin\omega\nu \mathrm{Re}\,\lambda_4 + \cos\omega\nu \mathrm{Im}\,\lambda_4]\}$, and $\nu = x_0 + \varepsilon\lambda_3$. Let the projection operator Π set a function subject to the orthogonality conditions (3.1) and (3.2) in correspondence with each odd periodic function from the space $H^s(0, L)$. It follows from the aforesaid that Eq. (4.1) is equivalent to the system

$$\boldsymbol{\lambda} = \boldsymbol{\lambda}_0 + \varepsilon\boldsymbol{\Lambda}(\mathbf{w}, \boldsymbol{\lambda}, \varepsilon), \quad \mathbf{w} = \varepsilon\mathbf{F}(\mathbf{w}, \boldsymbol{\lambda}, \varepsilon), \tag{4.9}$$

where $\boldsymbol{\lambda}_0 = (3a_0, 0, 0)$, $\boldsymbol{\Lambda} = (\Lambda_1, \Lambda_2, \Lambda_3)$, $\mathbf{F} = (F_1, F_2)$, and

$$F_1(\mathbf{w}, \boldsymbol{\lambda}, \varepsilon) = M\Pi f^+(\mathbf{w}, \boldsymbol{\lambda}, \varepsilon), \quad F_2 = \Pi_0 K f(\mathbf{w}, \boldsymbol{\lambda}, \varepsilon). \tag{4.10}$$

In deriving system (4.9), the fact that the shear operator commutes with any convolution operator was taken into consideration.

Below, by writing $\mathbf{w} \in V^s$ (correspondingly, $\mathbf{w} \in \mathring{V}^s$) we mean that $w^0 \in E^s (w^0 \in \mathring{E}^s)$ and $w^+ \in H^s(0, L)$, i.e., the norm $\|\mathbf{w}\|_{V^s} = \|w^0\|_{E^s} + \|w^+\|_{H^s(0,L)}$ is finite.

In other words, $\mathbf{w} \in V^s$ if $w = w^0 + \theta w^+ \in V^s$, and $\mathbf{w} \in \mathring{V}^s$ if $w \in \mathring{V}^s$.

Lemma. *Let $1/2 < s < 1$. There exists $\varepsilon_0 > 0$ such that, for $0 < \varepsilon < \varepsilon_0$, the operator $(\boldsymbol{\lambda}_0 + \varepsilon\boldsymbol{\Lambda}, \varepsilon\mathbf{F})$ transforms the sphere $B = \{\boldsymbol{\lambda} \in R^3,\ \mathbf{w} \in V^{s+1} \mid |\boldsymbol{\lambda} - \boldsymbol{\lambda}_0|^2 + \|\mathbf{w}\|^2_{V^{s+1}} < 1\}$ into itself and this operator is compressive.*

Proof. For brevity, it is expedient to adopt the following assumption: the nonlinear operator $\Phi(\mathbf{w},\boldsymbol{\lambda})$ belongs to the class Γ^r if, for all $(\mathbf{w},\boldsymbol{\lambda}) \in B$ and $(\mathbf{w}_1,\boldsymbol{\lambda}) \in B$, we have

$$\|\Phi(\mathbf{w},\boldsymbol{\lambda})\|_{V^r} + \sum_{j=1}^{3} \|\partial\Phi(\mathbf{w},\boldsymbol{\lambda})/\partial\lambda_j\|_{V^r} \leqslant C, \quad \|\Phi(\mathbf{w},\boldsymbol{\lambda}) - \Phi(\mathbf{w}_1,\boldsymbol{\lambda})\|_{V^r} \leqslant C\|\mathbf{w}-\mathbf{w}_1\|_{V^r}$$

with constant C. By writing $\boldsymbol{\Phi}(\mathbf{w},\boldsymbol{\lambda}) \in \Gamma^r$ we mean that all the components of the operator $\boldsymbol{\Phi}$ belong to the class Γ^r. Similarly, the functional $A(\mathbf{w},\boldsymbol{\lambda}) \in \Gamma^r$ if, for the indicated $\mathbf{w}$ and $\mathbf{w}_1$, we have

$$|A(\mathbf{w},\boldsymbol{\lambda})| + \sum_{j=1}^{3} |\partial A(\mathbf{w},\boldsymbol{\lambda})/\partial\lambda_j| \leqslant C, \quad |A(\mathbf{w},\boldsymbol{\lambda}) - A(\mathbf{w}_1,\boldsymbol{\lambda})| \leqslant C\|\mathbf{w}-\mathbf{w}_1\|_{V^r}.$$

The proof of the lemma reduces to checking the propositions $\Lambda_k \in \Gamma^{s+1}$ and $\mathbf{F} \in \Gamma^{s+1}$ $(1 \leqslant k \leqslant 3)$.

The functional $a(f)$ is linear and inequality (3.5) is true for this functional. If one sets $r = s-1$, it follows from the definition (4.5) of the functional Λ_0 and also from the properties (2.5) of the shear operator that $\Lambda_0 \in \Gamma^s$. Therefore, an operator that sets the pair (u,v) in correspondence with the vector $(\mathbf{w},\boldsymbol{\lambda})$ by formulas (4.3) and (4.6) belongs to the class Γ^s. The functions $f_0(u,v,\lambda_1,\varepsilon)$ and $f_1(u,v,\varepsilon)$ are entire analytic functions of their arguments. The composite functions $f_k(\mathbf{w},\boldsymbol{\lambda},\varepsilon)$, which correspond to them, also belong to the class Γ^s according to (2.1), (2.3), and (2.5).

For $s > 1/2$, the space V^s is continuously imbedded in C. Therefore, $|f(x)| \leqslant C\|f\|_{V^s}$ [6], and the integral in the right-hand side of (4.8) is defined for sufficiently small ε. In accordance with the estimates of composite functions (2.3) and (2.4), we have $\Lambda_1 \in \Gamma^{s+1}$. Since $b(f)$ is a linear functional, we have $\Lambda_4 \in \Gamma^{s+1}$ according to the estimate (3.6) for $r = s-1$ and the already established properties of the operator $f(\mathbf{w},\boldsymbol{\lambda},\varepsilon)$. Obviously, Λ_2 and Λ_3 belong to the class Γ^{s+1}. Because $f^+ = T_{-\nu}f^+ \in \Gamma^s$, according to the property (3.7) of the operator M we have $F_1(\mathbf{w},\boldsymbol{\lambda},\varepsilon) \in \Gamma^{s+1}$. Similarly, $F_2(\mathbf{w},\boldsymbol{\lambda},\varepsilon) \in \Gamma^{s+1}$ according to the same estimate. The lemma is proved.

Theorem. *For sufficiently small ε, there exists a solution of problem* (1.1)–(1.6) *such that $u - \varphi^0 \in \overset{\circ}{V}{}^{s+1}$ and $v - K_0\varphi^0 \in V^{s+1}$.*

Proof. The existence of a function u possessing the required properties follows from the principle of compressed mappings for system (4.9). Since $v = K_0 u$, the desired properties of the function v follow from (3.4).

Remark. For $\varepsilon = 0$ we have $u = \varphi$ and for $s \geqslant 1$ we have the function $\varphi^0 \in V^r (r < 1)$ and $\varphi^0 \notin V^s$. According to the theorem, the corrections to the solution associated with the nonlinearity are smoother than the fundamental solution.

This work was supported by the Russian Foundation for Fundamental Research (Grant No. 95-01-00859).

REFERENCES

1. J. J. Stoker, *Water Waves. The Mathematical Theory with Applications*, Wiley, New York (1992).
2. V. I. Nalimov, "Supercritical flows from beneath a shield," *Prikl. Mekh. Tekh. Fiz.*, **30**, No. 2, 77-81 (1989).
3. D. V. Maklakov, "Existence of a solution to the problem of a subcritical flow past a vortex," in: *Some Applications of Functional Analysis to Problems of Mathematical Physics* (Collected scientific papers) [in Russian], Inst. of Math., Sib. Div., Acad. of Sci. of the USSR (1990), pp. 89-102.
4. L. Hoermander, *Lectures on Nonlinear Hyperbolic Differential Equations*, Springer, Paris (1997).
5. L. Bers, F. John, and M. Schechter, "Partial differential equations," in: *Proc. of the Summer Seminar* (Boulder, Colorado, 1957), John Wiley and Sons, New York–London–Sydney (1964).
6. V. I. Nalimov, "Pseudodifferential operators with analytic symbols," *Sib. Mat. Zh.*, **38**, No. 3, 627-637 (1997).